Elasticity and optimization

Given the following demand function: $Q = a - bp$, where $a, b > 0$.

1. Calculate the elasticity and obtain the condition that p must meet for the demand to be elastic.
2. Suppose that $a = 10$ and $b = 2$ and now that a monopolistic firm faces this demand and an additional $W = 5 - 2p_w$, and its cost function is: $c = 5Q^2 + 2W^2$. Find the quantities that maximize profit, and the price set by this company for the two goods. (help: clear the price from the two demands and insert these expressions into the profit function).
3. Verify that the result found is a maximum with second-order conditions.

Solution

1. We calculate the elasticity

$$\frac{EQ}{Qp} = \left| \frac{\partial Q}{\partial p} \frac{p}{Q} \right| = b \frac{p}{a - bp}$$

For the demand to be elastic it must happen that:

$$b \frac{p}{a - bp} > 1$$

$$bp > a - bp$$

$$p > \frac{a}{2b}$$

2. First, we find the inverse demand: $Q = 10 - 2p$, we clear $p = 5 - Q/2$. With the other demand the same: $p_W = 5/2 - W/2$. We insert these into the profit function:

$$B = p_Q Q + p_W W - c$$

$$B = (5 - Q/2)Q + (5/2 - W/2)W - 5Q^2 - 2W^2$$

We derive to find the first-order condition:

$$B'_Q = 5 - Q - 10Q = 0$$

$$B'_W = 5/2 - W - 4W = 0$$

From the first expression, we clear:

$$Q = 5/11$$

From the second expression, we clear W:

$$W = 1/2$$

And with this, we find the price:

$$p_Q = 5 - (5/11)/3 = 5 - 5/9 = 10/3$$

$$p_W = 5/2 - W/2 = 5 - 1/4 = 9/4$$

3. We calculate the Hessian with the second derivatives:

$$B''_{QQ} = -1$$

$$B''_{WW} = -5$$

And the cross derivatives are equal to 0:

$$B''_{WQ} = B''_{QW} = 0$$

Therefore, the determinant of the Hessian is $|H| = -1 \times -5 = 5 > 0$ and as the second derivative with respect to Q is negative, we are facing a maximum.